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Understanding Fractal Analysis? The Case of Fractal Linguistics

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Abstract
Terms such as 'self-similarity', 'space filling', 'fractal dimension', and associated concepts have different meanings to different people depending on their background. We examine how methodology in fractal analysis is influenced by diverse definitions of fundamental concepts that lead to difficulties in understanding fundamental issues. The meaning of terms associated with fractal analysis needs to be clarified if this method is to be useful in diverse disciplines. It is our premise that communications that are result focused constitute a danger in perpetuating misconceptions of terms due to the concise nature of the writing and the reliance on references to fill in the procedural and conceptual gaps. Communicating effectively requires a sound understanding of the terminology and a clear and meaningful presentation. We address here communication and the nature of scientific discourse, '*fractal linguistics*'.

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Introduction

The literature shows that the fractal dimension (D) of an object reveals something about the natural world not otherwise apparent. This, combined with the seemingly simple procedures involved in fractal analysis, has led to the popularity of this procedure for pattern analysis. The interpretation of such analysis, however, is not always straightforward and confounding this are a number of misconceptions associated with terminology and meaning. This is particularly true for people interested in applications of fractal analysis but lacking a deep understanding of the underlying mathematical theory. In this paper we attempt to correct some of these misconceptions and suggest ways which will avoid their repetition. For language to have meaning in a community requires the specialists to take responsibility for the dissemination of knowledge that enables others to gain control of the discourse (subject matter) and contribute in a meaningful way. The lack of clarity can be attributed to people in specialized fields forming a specific linguistic boundary, *fractal literacy*, and expecting researchers in a different research field with different subject literacy to follow the discourse. The ideas underlying fractal analysis are inherently alien if compared, say, to the mathematics underlying simple principles in physics. The transfer of ideas from the theoretical domain of science to the applied is never easy but in the case of fractal analysis it appears to have been particularly difficult. The problem has been confounded in the case of fractals by the increasing specialization of science.

At the basis of this discussion is the fact that there appears to be little agreement on the meaning of the terms used in fractal analyses and, more importantly, how these terms are used in the context of describing the results of fractal analyses. As an example consider the question of whether biological forms are fractal [1, 2]. Strictly speaking, a fractal only describes forms that are strictly self-similar and infinite. Natural objects and their representations on computers are not fractals, yet are often so described. Are natural objects space filling? Like the term 'fractal', 'space filling' can have various meanings. With regard to fractal theory, space filling is an attribute of fractals and reflects that the recursive nature of the fractal tends to a space-filling limit. An example is the Peano curve that if drawn to the limit of infinity has infinite length and reaches every point of the delimited plane it is drawn on [3]. Biological forms do not have this property as they do not possess infinite length. Thus biological forms are not strictly space-filling. However, space filling can be used in a difference sense as discussed by Murray [1]. Here space filling is viewed as a process, a structure such as a plant is involved in, to optimize coverage.

Can we then use fractal analysis to discuss forms in nature? As the magnitude of published literature indicates, many seem to think this is possible. What is often not clearly stated is that 'fractals' can only be used as '*models*' for biological shapes because natural objects lack several characteristics that define fractals [3]. The most common characteristic cited for fractals is the exact repetition of detail at every observation scale, its *strict self-similarity* [4]. However, the construction of true fractal objects can be randomized. These objects possess *statistical* self-similarity and resemble more closely objects in nature. In the literature though, there has been a tendency to equate statistical self-similarity (which natural objects do possess to a differing degree) with strict self-similarity and to assume that estimates of statistical self-similarity are a test of fractality.

D is a parameter that describes the relationship between measured size and the measuring scale. Common examples from the literature that use D to describe relationships in space and time include heart rate irregularities, grazing effects on pastoral lands or structural attributes of blood vessel systems and neurons [5–10]. It is our observation, however, that conclusions drawn from fractal research remain at best tentative due to a lack of a generally comprehensible description of fractal theory and its relationship to the associated analysis procedures.

Concepts in fractal research such as the fractal dimension are not strictly defined and much of the terminology is used loosely [11–13]. Fractal research and discussion are characterized by the repetition of definitions and procedures that were initially ill defined and intended by the authors to be vague [7]. Thus, communicating effectively requires a sound understanding of the terminology and a clear and meaningful presentation. We address here communication and the nature of scientific discourse, '*fractal linguistics*'.

Communication

'A fractal set is a set in metric space for which the Hausdorff-Besicovitch dimension D is greater than the topological dimension D_T' [7, p. 361].

This sentence is the most often quoted citation found in journal articles to describe fractals, even though Mandelbrot stated, on the next page of his book, that this definition is rigorous, but also tentative [5]. The definition is an example of communication that may be mathematically rigorous, but uses language that is not accessible to non-specialists. Terms such as '*set*', '*metric space*', and '*Hausdorff-Besicovitch dimension*' are not very informative and meaningful to novices to fractal analysis and possibly to non-mathematicians, bearing in mind that defining what a dimension is, has involved the most brilliant mathematicians since the ancient Greeks [3].

How Can Fractal Analysis Be Better Understood?

Once a group of the scientific/professional community is established it becomes obvious that a type of professional socialization takes place. This socialization leads to communication barriers in terms of the discourse adopted by a specific group. Thus the scientific community in terms of research area and proficiency in fractal analysis can be divided into three main categories: (1) Those in the know who already possess a solid knowledge base in fractal and scaling theory [7, 14, 15]. Thus the use of specific language by a group establishes an identity for this group with a specialist domain of knowledge and expertise. (2) Those with a solid knowledge base in other disciplines and an understanding of mathematics or physics [3, 16–18]; those that bridge the gap between (1) and (2), that is, those that develop tools for fractal analyses and aim at describing these clearly [19]. Category 3 should primarily consist of category 1 researchers that aim to make the theoretical basis more accessible to applied research. Despite the efforts of the third group the gap between groups 1 and 2 is still considerable. In order to have control of the ways language is used to create meaning in fractal analysis throughout the community specialists are necessary to take responsibility for the dissemination of knowledge that enables others to gain control of the discourse (subject matter) and contribute in a meaningful way. The lack of clarity lies in the fact that many in the first category, being entrenched within a specific linguistic boundary, *fractal literacy*, find it hard to communicate the necessary information to researchers in a different research field with different subject literacy. Papers in the second category reflect that the authors can acquire the necessary literacy and know how fractal analysis is applied but perpetuate that same inaccuracy in terminology. Thus, it is this section where inherent uncertainties in definitions are most easily propagated. One likely reason for this is that the communicator adopts the new language base but may not have a correct understanding of the terms. Should a correct understanding exist then it is important to communicate this explicitly. This latter requirement is important as many researchers in specialized fields do not have familiarity with expertise outside their special interests yet are, in many instances, referred to publications from the first category above. The third category is meant to provide the tools and structure necessary to bridge the gap between the first and the second category. This third category is not well presented [3, 20–22]. It is our assertion therefore that category 1 and 2 researchers have an obligation to disseminate their work in a manner understandable to all in addition to their publications in specialist journals. These communications certainly exist. Indeed Mandelbrot [23], who is known as the father of fractal geometry and a mathematician, published a paper that is aimed for the uninitiated, as did Jürgens et al. [24], albeit this latter paper is more complicated. Yet, why are the most active uses of fractal analysis in physics? [25]. The problems which surround fractal research can be illustrated by the case of diffusion-limited aggregates (DLAs). These are the result of a random growth process and could serve as a model for many such processes, for example, in biology. Most of the analysis of DLAs has been undertaken by physicists who have a good grasp of the theory. The inherent complexities of DLAs, however, have meant that there is considerable disagreement about the interpretation of these objects. Is a DLA self-similar? [25–27]. Is a DLA a multifractal? [26, 28–30]. These publications and their authors are well known in the field of physics and thus their results are widely cited. All papers cited here involved physicists and apart from one paper were published in physics journals and written for physicists. This may explain why fractal analysis has not been as successful in the biological sciences as it has in engineering and physics. What is lacking in the field of fractal analyses is for researchers in the field of fractal theory to articulate clearly how they themselves learned both theory and application, *the fractal discourse.* It is important to disseminate to the rest of the community how meaning is constructed and communicated in their field.

Fractal Linguistics

The popularization of fractals and chaos has led to widespread interest in applying these ideas in a range of scientific disciplines. What is not immediately apparent to scientists entering this field is that there is no consensus on a number of the most basic issues, even amongst experts. Terms, such as those mentioned at the beginning of this paper, and procedures used in fractal analysis need to be understood before fractal analysis becomes an effective tool. It is the World Wide Web that can be of use here as many sites exist that address '*basic*' issues. We exemplify our stance in the next section: teaching the skills of fractal literacy: a language base for novices.

A Language Base for Novices

This section cannot cover all the terms that are required for a good language base. It rather aims at correcting some common misconceptions. What then is a fractal and how is fractal analysis performed? We start by describing some of the properties displayed by fractals. Fractals do not have inherently smooth surfaces no matter what magnification is used to examine the object.

How Can a Fractal Be Described?

A common, yet 'mathematically' inappropriate term used to describe the surface of objects is their '*roughness*'. The term roughness is used in common language to describe what is, in fractal linguistics, an object's space-filling capacity or surface irregularity. Yet, roughness and fractal dimension do not describe the same feature. Roughness is a measure of the average variation about a mean and is not related to scale or changes in scale of measurement. D quantifies the variation in length, area or volume with changes in the size of the measuring scale [16].

How can the attributes of fractals and natural objects be described more appropriately? Three of these are *characteristic length, self-similarity* and *complexity*. Any non-fractal form can be approximated by a simple shape with the same characteristic length. Thus, a sphere can approximate the Earth. The existence of a characteristic implies a smooth surface. In case of the Earth, the highest mountain is much smaller than the diameter. Another way of looking at this is to say that the surface of the Earth becomes smoother as the magnification is decreased. Fractal objects cannot be represented by any combination of shapes with characteristic lengths. A cloud is such an object. Thus for many forms, Euclidean geometry and its associated Euclidean dimension suffice to characterize these. Ideal fractal objects have no characteristic length. Consider the representation of several steps in the construction of a Koch curve (fig. 1). Even at the fourth step of con-

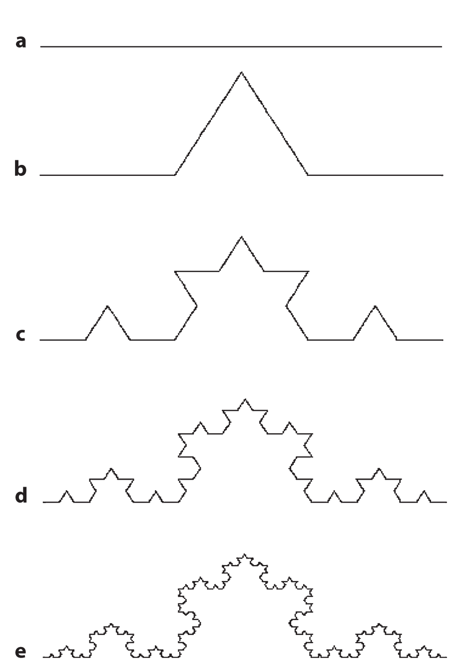

Fig. 1. a–e The Koch curve displaying the iteration process over several generations.

struction, it becomes apparent that no characteristic scale exists for the Koch curve and that it can only be approximated by the use of a number of spheres with different sizes and not by one simple shape.

Figure 1 also demonstrates the self-similarity of fractal images as any change in scale will show more detail as magnification is increased but will not lose detail if the magnification is decreased. Fractal analysis is then the procedure that compares the size of the outline or mass of the object for each scale of the measurement. This can be done using disks or squares. If this change of size is constant with change of scale on a double logarithmic plot, the form approximates a fractal and the gradient of the line through these points is proportional to the fractal (self-similarity) dimension (fig. 2).

The gradient is mathematically defined by

$$D = \frac{\log N(e)}{\log(1/e)},$$

which transforms into $N(e)^D = 1$, a power law relationship, where the exponent '*D*' is the fractal dimension (for the Koch curve in fig. 2, $D = 1.246$). The mathematical definition of the fractal states that the size of the image scales indefinitely by a constant proportion. The fractional part of the exponent indicates the complexity of the object. If the fractional part of the exponent is 0, *D* would be equal to 1 and equal to the Euclidean dimension for a line. If the line becomes more space filling the fractional part of the exponent increases towards 2.

Complexity is another common term used to describe the surface irregularities or the intricacy of a branching structure. *D* indicates whether the structure contains a degree of self-similarity, which in turn can be an indication of the underlying biological process that leads to the observed pattern. Self-similarity, as Mandelbrot [7] points out, is a simple design principle that can be independent of genetic determination and only dependent on a consecutive scaling with changes of magnitude. The dimension exponent can indicate how a structure's branches scale from the parent to the daughter branches as can be observed in the lung bronchi for instance [8]. *D* can also be an indicator of how space filling a structure is. If *D* has a value of 1.2 then the structure is not as space filling as if *D* was 1.4. Thus the feature that makes fractal analysis interesting is measuring the range of the statistical self-similarity across scaling levels and the associated estimate of *D*. Statistical self-similarity can be an indicator of a growth process in tubular structures such as blood vessels, lung bronchi or neurons. The fractal dimension, provided the structure displays 'statistical self-similarity', can be used to infer a physiological function such as flow of air in lungs or electrotonic properties in neurons as the diameter exponent in this special case equals the fractal dimension in some cases [7].

Accuracy or precision is important in fractal analysis as with any other measurement. Using Euclidean objects is the easiest way forward as the dimension of these is known. However, because we want to ascertain the fractal dimension of fractal-like structures it is important to use test images that are similar to the structures to be investigated. In general *fractal* objects such a as the Koch snow flake or a DLA is used for such calibration [25]. However, a fractal object is the result of repeated transformations of a geometrical figure that lead to a strictly self-similar pattern with no scaling limit. In practice only prefractals can be used to calibrate the measurement system. These are approximations to true fractals – examples of true fractals shown at a finite number of iterations – since truly infinite cases are not possible in the real world. The '*Koch curve*' shown in figure 1 is not a true fractal, as none of the iteration levels depict the case of infinity.

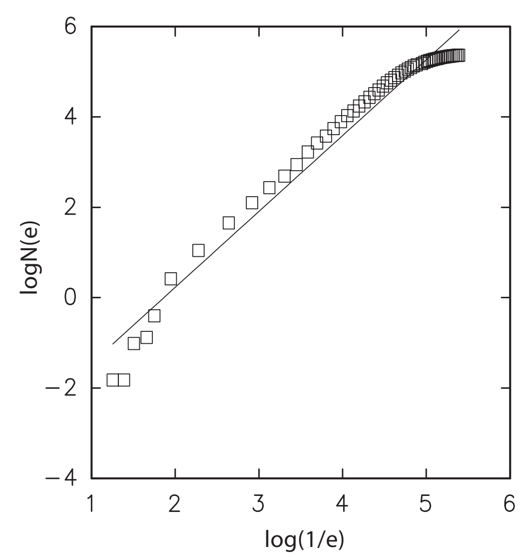

Fig. 2. Double logarithmic relationship between the measuring scale (1/e) and the size of the image *N*(e). The regression line through the data points indicates the range of statistical self-similarity and its gradient is proportional to the scaling exponent. Method of applying mass-radius analyses to quantify the branching complexity of a fungal colony. The mean of 5 separate analyses of this image, using different centres of origin, gave a mean Df = 1.691 ± 0.017, $r^2 = 0.967$.

Any incomplete representation of an ideal fractal, as is the case with computer representations, is referred to as prefractal [31]. In practice when prefractals are used as test images, accuracy in the strict sense cannot be ascertained as the image is not completely represented on the screen. However analytical estimates using many fractal analysis algorithms agree with the theoretical dimension value.

Once a biological object is analyzed and a constant slope is obtained for the log-log representation of scale versus size as a function of scale, the object is said to be self-similar with a certain fractal dimension. This broad use of the term *self-similar* has led to biological forms being often identified as fractal. Biological forms are not fractals, as they are not characterized by identical patterns at different scales (strict or linear self-similarity) [7, 32, 33]. Biological forms are at best statistically self-similar over a limited range [32]. This misconception may have arisen from the use of the term self-similarity and scale invariance to indicate both strict self-similarity for ideal fractals and statistical self-similarity if biological forms are described [3, 34–36]. Similarly, obtaining D for an object does not indicate that this object is fractal. Fractality is determined by the relationship between the *observation scale* and the *measured size* of the object, which has to be constant without a limit. Once some of the terminology becomes clear, the application of the algorithms still remains a mystery.

The Black Box

Prior to the advent of fractal analysis, the relationship between scale and length of coastlines was discussed by Richardson [37]. He used callipers of different sizes (scale) to determine the length of the coast of Britain.

What he found was that the coastline length increased as the scale was decreased according to a power law relationship. The *exponent* is then the parameter that quantifies this relationship. Thus, whether the object is fractal or not did not have any bearing on the aim of the analysis [37–40]. Currently many different methods such as box counting, dilation, mass radius as well as the calliper method used by Richardson are used to determine similar relationships. These methods are all analytical tools that estimate the relationship between a scale of measurement and the size/mass of the object being measured. This is now referred to as fractal analysis and the exponent is the fractal dimension. What is a dimension? There exists disagreement among physicists and mathematicians about what constitutes a dimension and what measures can be included in this term. Apart from the Hausdorff dimension other so-called fractal dimensions such as the Minkowski and the Kolmogorov dimension are not really dimensions in a mathematical sense [3, 7, 16, 41]. However, in the literature authors often refer to these two measures as dimensions possibly because of trying to simplify what is really quite a complex area in mathematics. Analytical tools, such as the dilation method and box-counting method, are based on the Minkowski measure and Kolmogorov measure, respectively. These two measures are in themselves approximations of the mathematically rigorously defined fractal dimension (Haudorff dimension) and are only equal for strictly self-similar objects [21, 35]. Indeed books and articles containing the terms 'fractal analysis' in their titles often do not point out that a strict application of the rules of mathematics to the procedures of fractal analysis is not possible. One only needs to observe the intricate definition of the Hausdorff dimension, which is not usable in practice and the shortcomings of the analysis procedures in estimating the Hausdorff dimension using for instance the box-counting method [32, 41, 42].

Having decided to undertake fractal analysis, we need to consider several procedural steps. The first question to ask is whether or not it is important that the image is fractal. No! Fractal analysis procedures are applicable to non-fractal objects (which all biological objects are). Next is the question of what aspect of the structure is to be analyzed, especially with respect to the necessity of representing the structure as a 2-dimensional (2D) object on the computer screen. Is a surface the same as a boundary or border? For instance, when considering a structure such as a particle aggregate, many authors have a different idea of what a 2D image versus a 3D object is or what a surface, a boundary or a border is. What is mass in 2D [41, 43]? What does it mean to analyze a surface when some people see a surface like a relief map, while others view it as the boundary or perimeter of a 3D object? Understanding these terms is critical as techniques for analyzing the scaling characteristics are very different [44–48]. Image silhouettes, for example, have been used in the analyses of fractal-like characteristics in sludge aggregates by several investigators [47–49]. These silhouettes can be presented as sectioned boundaries or as the boundary of the silhouette. Estimates of the dimension of silhouette boundaries are consistently smaller than those of sectioned boundaries [48]. Associated with this is the significant problem of determining the 'appropriate' fractal dimension. What type of fractal does the image represent? Pfeifer [44] suggests several 'prototypes' of fractals including mass and porous fractals. This is important as different fractal analysis procedures have been suggested, depending on the prototype the object is akin to. Note, though that there is no agreement with this step and some authors use the boundary of an object regardless if its length or mass with respect to scale is investigated [50–52].

Having decided that the image to be analyzed does not have to be fractal but that some type of scaling rule may be implicit in the morphology of the structure, the next step is to decide on the type of algorithm to be used [44, 53]. Now it becomes important to extend our working definition of self-similarity to include the broader concept of *self-affinity*. Self-affine images scale differently in the *x* and *y* directions. When the object scales equally in the *x* and *y* direction, self-affinity is the same as self-similarity [3, 54].

Before starting the analysis, some image manipulation or preparation of the sample, such as sectioning a 3D object versus representing it as a silhouette, may be required and may have an effect on the estimate of *D* [5, 43]. The image needs to be available for the specific fractal analysis software for analysis. Here the placement of the image and the size of the image within the acquisition screen may also influence the estimated *D*. Many of the fractal analysis applications then include a choice of the number of starting points for the box placement and selection of the range of box sizes and whether the image is to be rotated [4, 5, 55–57]. These options need to be investigated with all software as both the extent of statistical self-similarity and the magnitude of *D* are influenced by these procedures. Further, application of conventional fractal techniques such as box-counting are inappropriate for analyzing self-affine fractals [58]. However, using a section parallel to the nominal surface orientation, the resulting boundary lines may be statistically self-similar or scale-invariant. These boundary lines can then be analyzed to characterize the surface. The boundary of the silhouette is an approximation of this sectional boundary as discussed previously.

Using 'fractal' test images to ascertain the efficacy of the procedure may also be misleading because of how this aspect of fractal analysis is often described. The accuracy of fractal analysis methods is by its very nature difficult to determine. Consider the determination of *D* for a branching fungus. The control image needs to be of similar shape to the image tested [59]. DLAs are often chosen for this task [25, 52]. Yet physicists are not in agreement that a DLA is a fractal or what its exact dimension is [23, 33, 43]. If the dimension of a boundary has to be determined, such as for a particle aggregate, the *Koch curve* can be used as a control image. The precise *D* for this curve is 1.264 but as the curve cannot be represented accurately, that is, with an infinite amount of detail, the estimate of *D* will in many cases deviate from the theoretical. This can be avoided if a level 4 or 5 representation of the curve is used (see fig. 1). In addition there is a pixelation/staircase effect that is dependent on the size of the pixels (resolution of the screen) that influences the complexity of the border. In classification tasks for instance, the robustness of the procedure is important and this can even be tested by using Euclidean images such as a square or circle. The question is not whether the images are fractal but if there is a difference between images in their scaling behaviour [60, 61].

What fractal analysis procedure is appropriate has been briefly mentioned above. Despite arguments in the literature that both mass and boundary fractal dimensions should be measured using image boundaries, investigators have often used these fractal analysis techniques to measure different types of images [5, 18]. Others used the box-counting technique on silhouette areas to characterize the mass of flocs and granules [48]. The resulting dimension, though perhaps not a theoretical mass fractal dimension, was found to be effective in differentiation tasks [43]. Fractal analysis methods such as the dilation technique may also be applied to image boundaries (including both perimeters and pore space boundaries) of a 2D image [37, 49]. However, the resulting fractal dimension is greater than the fractal dimension of the perimeter as the pore boundaries provide an increase in the space-filling ability of the surface.

Once the data has been obtained as a scaling relationship between the logarithm of the scale of observation and the logarithm of the size of the image, the extent of statistical self-similarity and the fractal dimension can be determined. A decision that needs to be made here is whether to include all data points or whether there is one, two or more clearly defined linear segments apparent in the plot [3, 6, 20, 51]. Inherent in the linearity of the plot is the selection of the different scales of the measuring device such as a box or circle. If the measuring scale is very much larger than the size of the object, the box always covers the object and the number of boxes required to cover the object, regardless of the size, will always be 1 and the slope or *D* will be 1. Alternatively, if the size of the box is smaller than the 'line width' of the object, then one is measuring the area of a solid object and the slope or *D* is 2 [62]. Having chosen the appropriate size for the boxes, the log-log plots that represent the scaling characteristics of biological forms still do not show a constant relationship between the scale of measurement and the size/mass of the object. The points where this occurs are the upper and lower cutoff points between which statistical self-similarity occurs. One can remove data points one at a time until the squared correlation coefficient approaches some previously defined number (e.g. 0.995), or utilize a combination of curve-fitting tests and curvilinearity of residuals to identify the largest range over which the image displays statistical self-similarity [22, 63, 64]. Curvilinearity can be tested by fitting the data to first and second order polynomials and comparing using a χ^2 test the better fit or use the error in Y [22, 63]. If regression analysis is used, one needs to consider that converting linear data to log-log data changes the profile of the distribution of the data points. This can be corrected for by weighting the log-log data points [65]. However, any method involving linear regression may not be suited to measure complexity and an alternative may be more appropriate [41]. The final outcome is an estimate of the fractal dimension for a biological object and an indication of the range of statistical self-similarity.

The above considerations have attempted to provide a start to sound understanding of fractal analysis in that they clarify the meaning behind terms commonly used in this field and aspects of the procedure.

Conclusion

Being familiar and understanding the basic terms used in any analysis procedure set up a linguistic domain that is imperative to ensure that the results are meaningful. This paper concentrated on several that are relevant to fractal analysis. The process of selecting an appropriate fractal analysis technique with reference to image characteristics was also outlined. This then represents together with the section on 'language base for the novice' and 'the black box' an attempt to make this methodology available and understandable to scientists within the field of fractal analysis and in other fields of research.

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